APERIODIC MOTION OF A SUSPENDED MAGNET

BY

G. E. MARSH

ARMOUR INSTITUTE OF TECHNOLOGY

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The Aperiodic Motion
# of a
Suspended Magnet.

A Thesis

Presented by

George E. Marsh

to the

President and Faculty

of

Armour Institute of Technology

For the Degree of

Electrical Engineer.

Approved
E. R. Freeman
Prof. of Elect Eng

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L. C. Monin
Dean of Cult. Studies

The Oscillations of a Damped Magnet.

It is more than a coincidence that the same relationships between fundamental concepts are found in different physical sciences. In each science there are concepts or characteristics whose function and action are similar to those of certain quantities in a related branch. The one is the analogue of the other. In mechanics there are such elements as mass, motion, momentum, etc, and whose counterparts are recognized, under various names, in the cognate sciences. Between physical magnitudes of the same fundamental character, in the last analysis of cause and effect, we find a similarity in function and result.

The relationship with which the present paper is concerned is of the foregoing type and it is, in fact, an expression for physical phenomena in both

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mechanics and electricity. Consequently, all of equations and results that are herein developed and explained are available for direct use in any one of several distinct cases, each of which may be taken to be the counterpart of the present one.

In the course of our investigations, a number of somewhat complex expressions and relationships are encountered and in order that their origin and intrepretation may be obvious, and their applications may be easily understandable, it will be desirable to approach the subject in a very simple and easy manner, as we shall now proceed to do.

When a body, capable of moving about a fixed axis, is displaced from its position of equilibrium it is, in general, subjected to the action of a couple tending to return it to its initial position. Following the displacement, the body usually executes

vibrations about its position of equilibrium and, in all material cases, gradually comes to rest there. This type of motion is represented by the oscillations of a pendulum, of a suspended magnet in a magnetic field, of a balance-wheel, of a mass suspended by an elastic wire, etc. In the theoretical consideration of any of these cases there may or may not be damping action arising from frictional resistance of one kind or another or from divers reactions.

The same equations and results, obtained from a consideration of any of these examples will apply, with proper interpretation, to an electric circuit containing resistance, capacity and inductance and a current whose value is a function of the time.

In the above illustrations there is a couple tending to restore the body to the position of equilibrium and which is a function of the angular dis-

placement from the position of rest at ant instant. This couple may arise, for example, from the presence of an external force, such as gravitational attraction, the earth's magnetic field, or thru the elastic or torsional property of the suspension, or from any combination of them. In all cases referring to a practicable system there are damping forces called into being by the velocity and which, therfore, are some function of it.

The equation of motion of the body may be written at once and in accordance with d'Alembert's Principle, a mathematical statement of Newton's Third Law on Motion. It is--- the impressed forces together with the forces of inertia form a system of equilibrium. The former comprise the restoring forces and those arising from the resisting or damping agencies. In the case of a rigid body capable

of oscillating about an axis, the forces of inertia are given by the product of the moment of inertia and the angular acceleration, and are represented by $K\frac{d^2s}{dt^2}$. The directive forces, responsible for the return of the body to the initial position may be represented by $-Df_2(s)$ and the energy-consuming, or damping forces by $-Pf_1(\frac{ds}{dt})$, D and P being constants.

The equation of motion is therefore

$$K\frac{d^2s}{dt^2} + Pf_1(\frac{ds}{dt}) + Df_2(s) = 0. \qquad (1).$$

This is the expression when the origin coincides with the position of equilibrium, the zero position, as it will be frequently designated. If, however, these two points do not coincide, but are separated a distance ε, the equation becomes

$$K\frac{d^2s}{dt^2} + Pf_1(\frac{ds}{dt}) + Df_2(s-\varepsilon) = 0, \qquad (2).$$

The solution of either of these forms is dependent on a knowledge of the nature of the functions

$f_1(\ )$ and $f_2(\ )$ and in general it is only when they are simple in relationship that the problem can be successfully handled. To facilitate the integration, $f_2(\ )$ is assumed to be represented by $\sin S$ and $f_1(\ )$ by the first or second power of the velocity. Such assumptions as these correctly represent the circumstances of a magnet supported by a torsionless fiber and oscillating in a magnetic field, the air damping being proportional to the velocity. The equation is $K\frac{d^2S}{dt^2}+P\frac{dS}{dt}+D\sin S=0$. (3).

If the body execute oscillations so small that the sine of the angle is substantially equal to the angle itself, or if a non-magnetic body be suspended by an elastic fiber, then the equation takes the form

$$K\frac{d^2S}{dt^2}+\rho\frac{dS}{dt}+DS=0 \qquad (4).$$

or

$$K\frac{d^2S}{dt^2}+\rho\frac{dS}{dt}+D(S-\epsilon)=0, \qquad (5)$$

corresponding to Eq.(2).

In the general Eq. (4), it is to be understood that K is the moment of inertia of the suspended body, P the damping constant, that is, the factor which multiplied into the angular velocity of the body gives the resisting moment of the motion, and D is the restoring force and defined as the constant relation between the moment, arising from a small displacement and the angular magnitude of the displacement. If the restoring moment arise from the torsional rigidity of the suspension, then $D = \frac{1}{2l} \cdot \pi \phi r^4$ where ϕ is the modulus of torsional elasticity, r is the radius of the suspension and l its length.

It is to investigate the cases arising under this equation that the present paper is devoted. But before doing so it will not be without value to review the general ideas involved in undamped oscillatory motion and to that end the next few pages will

be given over to the consideration of a pendulum swinging in a non-resisting medium.

Chapter 1.

As there is no damping, Eq.(4) becomes

$$K\frac{d^2s}{dt^2} + \mathcal{D}s = 0, \qquad (6).$$

and its statement may be made as follows: the pendulum possesses a variable motion and an acceleration that is directly proportional to the displacement. In particular, we have

$$ml^2\frac{d^2s}{dt^2} + mlgs = 0, \qquad (7).$$

where l is the length of the pendulum, m is the mass of the bob and g is the gravitational constant.

If the equation above be rewritten, for convenient manipulation, the changes being easily understood, as

$$\frac{d^2s}{dt^2} + f^2s = 0 \qquad (8).$$

or as

$$\frac{d^2s}{dt^2} + f^2(s-\varepsilon) = 0 \qquad (9).$$

the solution is easily found to be

$$s = B\sin ft - C\cos ft, \qquad (10).$$

where B and C are arbitrary constants introduced thru integration. Let $B = A\cos\alpha$ and $C = A\sin\alpha$ where A and α are also constants, then Eq.(10) becomes

$$\begin{aligned} S &= A\sin(ft-\alpha) \qquad (11). \\ &= A\cos(ft-\alpha-\pi/2) \end{aligned}$$

or

$$S = \varepsilon + A\sin(ft-\alpha)$$

if Eq.(9) be accepted as representing the actual motion. By inspection it can be seen that α is the angular position,s, of the bob at the time $t = 0$; that is, the origin of time, in the general case, is not coincident with the passage of the body thru zero, the position of equilibrium.

It is seen that if the angle $(ft-\alpha)$ be increased by 2π the values of the swing are the same. Hence the motion passes thru a cycle, repeating itself at equal times, and is accordingly periodic. The maximum displacement is A and is attained when $(ft-\alpha)$

is equal to $\pi/2$ or an odd multiple thereof. Accordingly, as the maximum value of the swings remains the same with the lapse of time, the motion shows itself to be without damping and thereby confirms the physics of the case.

If the times of two consecutive elongations, or maximum displacements, on the same side of the origin be designated bt t_1 and t_2 and as the angles at which these positions are attained differ by 2π, there results, the time-interval of a complete oscillation being designated by T, $(t_1 f - \alpha) = \pi/2$, say, and hence $(t_2 f - \alpha) = \frac{5\pi}{2}$, or $T = (t_2 - t_1) = 2\pi f$. (11^2)

Let us now determine the values of the constants A and α, and in the most general manner, that is, when $s = s_1$, $ds/dt = v_1$ and $t = t_1$. It is easily found that

$$\alpha = \frac{2\pi t_1}{T} - \tan^{-1}\frac{2\pi S_1}{V_1 T}, \qquad (12).$$

and

$$A = \sqrt{S_1^2 + \frac{V_1^2 T^2}{4\pi^2}}. \qquad (13).$$

In the special case when $s = a$, $ds/dt = S' = 0$ and $t = 0$, we find $\alpha = -\pi/2$ and $A = a$. When $s = a$, $ds/dt = s' = 0$ and $t = T/4$, we find $\alpha = 0$ and $A = a$. In the first case t is estimated from the instant of greatest elongation and in the second case from the transit of the zero position.

The introduction of T from Eq.(11) into Eq. (10) gives the displacement in the form

$$S = A \sin\left(\frac{2\pi t}{T} - \alpha\right). \qquad (14).$$

The velocity is $$S' = \frac{ds}{dt} = \frac{2\pi A}{T} \cos\left(\frac{2\pi t}{T} - \alpha\right), \qquad (15).$$

and the acceleration $$s'' = \frac{d^2 s}{dt^2} = -\frac{4\pi^2 A}{T^2} \sin\left(\frac{2\pi t}{T} - \alpha\right). \qquad (16).$$

If the constants from Eqs.(12) and (13) be introduced into the last three expressions we have

$$S = \sqrt{S_1^2 + \frac{v_1^2 T^2}{4\pi^2}} \cdot \sin\left(\frac{2\pi t}{T} - \frac{2\pi t_1}{T} + \tan^{-1}\frac{2\pi S_1}{v_1 T}\right), \quad (14').$$

$$S' = \frac{2\pi}{T}\sqrt{S_1^2 + \frac{v_1 T^2}{4\pi^2}} \cdot \cos\left(\frac{2\pi t}{T} - \frac{2\pi t_1}{T} + \tan^{-1}\frac{2\pi S_1}{v_1 T}\right), \quad (15').$$

$$-S'' = \frac{4\pi^2}{T^2}\sqrt{S_1^2 + \frac{v_1 T^2}{4\pi^2}} \sin\left(\frac{2\pi t}{T} - \frac{2\pi t_1}{T} + \tan^{-1}\frac{2\pi S_1}{v_1 T}\right). (16').$$

It is to be noted that, as a matter of simplification, it will occasionally be convenient to use s' and s" to indicate the first and second derivatives of the displacement S.

If the initial conditions be specified as $s = 0$, $s' = v_0$ and $t = 0$, then Eq.(14) becomes

$$S = \frac{v_0 T}{2\pi} \sin \frac{2\pi t}{T} . \qquad (17).$$

The velocity is $S' = v_0 \cos \frac{2\pi t}{T}$. (17')

The maximum elongagion has a value of $Tv_0/2\pi$.

The maximum and minimum values of the fundamental magnitudes that have been shown in explicit relations in the above equations, together with their times of occurance, are given in the adjoining table.

	Max. Value.	$t_{max.}$	Min. Value.	$t_{min.}$
Angular Displacement.	A	$T(\frac{n}{4} + \frac{\alpha}{2\pi})$	0	$T(\frac{m}{2} + \frac{\alpha}{2\pi})$
Angular Velocity.	$2\pi A/T$	$T(\frac{m}{2} + \frac{\alpha}{2\pi})$	0	$T(\frac{n}{4} + \frac{\alpha}{2\pi})$
Angular Acceleration.	$-2\pi A/T$	$T(\frac{n}{4} + \frac{\alpha}{2\pi})$	0	$T(\frac{m}{2} + \frac{\alpha}{2\pi})$

In the table m and n indicate and positive, even or odd whole numbers respectively.

Our attention will now be transferred to the determination of the periodic time of vibration and to that end let the origin be located at an elongation and t be estimated from the instant that $s' = 0$. If the arc s be described by the body in the time t', and if the arc $A =$ arc $s +$ arc θ, and l the radius of the arc, then we have $s = l(A - \theta)$. Since the velocity at t' is that which would be acquired by the body in falling vertically thru the distance between parallel horizontal lines intercepting the ends of the arc s, there results from $V^2 = 2gh$, the well-known gravitational formula,

$$\frac{ds^2}{dt^2} = 2gl(\cos\theta - \cos A) \tag{18}$$

g being the usual gravitational constant. This changes to, thru the medium of $s = l(A - \theta)$,

$$\frac{d\theta^2}{dt^2} = \frac{2g}{l}(\cos\theta - \cos A) \qquad (19).$$

$$= \frac{4g}{l}(\sin^2 A/2 - \sin^2 \theta/2) \qquad (20).$$

and hence $\frac{d\theta}{dt} = \pm 2\sqrt{\frac{g}{l}}\sqrt{\sin^2 A/2 - \sin^2 \theta/2}$ (21).

It is to be noted that cosθ diminishes as t increases, dθ/dt is therefore negative, and hence Eq.(21) is

$$\frac{d\theta}{dt} = -2\sqrt{g/l}\sqrt{\sin^2 A/2 - \sin^2 \theta/2} \qquad (22).$$

Accordingly, $2\sqrt{g/l}\int_0^{T/4} dt = +\int_0^A \frac{d\theta}{\sqrt{\sin^2 A/2 - \sin^2 \theta/2}}$ (23).

or $2T\sqrt{g/l} =$ (23').

If sinA/2·sin ϕ be substituted for sinθ/2, a change that is valid as θ is always less than A, there results

$$T = 2\sqrt{l/g}\cdot 2\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 + \sin^2 A/2 \sin^2\theta}}$$

or, as sin A/2 is a constant, $= 4\sqrt{l/g}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - K^2\sin^2\theta}}$ (24).

The denominator is expressible as a series, viz.

$$1 + \frac{1}{2}K^2\sin^2\phi + \frac{1}{2}\cdot\frac{3}{4}K^4\sin^4\phi + \cdots \qquad (25)$$

and from the formula $\int_0^{\pi/2} \sin^{2n}\phi\, d\phi = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots (2n)}\cdot\frac{\pi}{2}$ (26).

there results $T = 2\pi\sqrt{l/g}\left\{1 + \left(\frac{1}{2}\right)^2 K^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 K^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 K^6 + \cdots\right\}$ (27)

This expression is the time of a complete

oscillation of an angular extent 2A. In case A is so small that its square and all higher powers can be neglected, that is, the body executes only small excursions, then our last equation becomes

$T = 2\pi\sqrt{l/g}$ or by Eqs. (7) and (8) (28)

$$\begin{aligned} T &= 2\pi\sqrt{K/D} \\ &= 2\pi/f \end{aligned} \qquad (29).$$

Chapter II.

The preceeding pages provide a cursory introduction to the subject-proper and it is to this that we shall now give our attention. To that which has gone before, a new condition is added, namely that of damping, and to secure an equation that actually represents a system that is materially possible, we shall assume that the damping effect is proportional to the first power of the velocity. Eq.(4) may be written as

$$\frac{d^2s}{dt^2}+2n\frac{ds}{dt}+f^2s=0 \qquad (30)$$

or as

$$\frac{d^2s}{dt^2}+2n\frac{ds}{dt}+f^2(s-\varepsilon)=0$$

if the origin be not at the position of equilibrium. Inasmuch as we can always move the origin into coincidence with the zero position, thereby simplifying the results, we shall assume that this has been done, and from this time on, use the corresponding form of equation.

This equation occurs frequently and in varied garb in mathematical physics in many cases of damped motion, such as the oscillations of a pendulum in a resisting medium, of a d'Arsonval galvanometer coil, or of the moving system of a Thomson galvanometer or a magnetometer, or of the motion of a current or a charge in an electric circuit, etc.

It is to be noted that the equation refers to a system of unit moment of inertia and that the coefficient of damping is 2n and that the coefficient of restitution is f^2.

The solution of this equation is possible on the following principle:- any solution of the equation multiplied by a constant is a solution and the sum of two solutions is a solution. To obtain particular solutions, let $s = e^{mt}$, $s' = me^{mt}$, $s'' = m^2 e^{mt}$, and hence Eq.(30) becomes

$$m^2 e^{mt} + 2nme^{mt} + f^2 e^{mt} = 0,$$

and $$m = -n \pm \sqrt{n^2 - f^2}.$$

If the two values of m be designated by m_1 and m_2, then the particular solutions are

$$S = e^{m_1 t} = e^{-n - \sqrt{n^2 - f^2}},$$
$$S = e^{m_2 t} = e^{-n + \sqrt{n^2 - f^2}},$$

and the general solution is given by

$$S = Ae^{m_1 t} + Be^{m_2 t},$$
$$S = Ae^{-(n + \sqrt{n^2 - f^2})t} + Be^{-(n - \sqrt{n^2 - f^2})t}. \quad (31).$$

There are three separate cases in the detailed consideration of Eq. (31) and they are determined by the value of the radical, that is, by the relative magnitude of n and f. In general, then, when m_1 and m_2 are real quantities and unequal, the solution is of the form

$$S = Ae^{m_1 t} + Be^{m_2 t}, \qquad (32).$$

and of the form

$$S = (A + Bt)e^{-nt}, \qquad (33)$$

when the roots are equal. If the roots be imaginary,

$$S = e^{-nt}(A \sin rt + B \cos rt). \qquad (34).$$

It shall be our problem to consider in some

detail the characteristics of these equations under various assumptions and to that end we shall divide the investigation into three parts.

Part I, $n < f$. Eq.(34).
Part II, $n > f$. Eq.(32).
Part III, $n = f$. Eq.(33).

Part I.

The radical is imaginary and if the two roots m_1 and m_2 be written $m_1 = -n - ig = -n - r$; $m_2 = -n + ig = -n + r$; where $i = \sqrt{-1}$, $g = \sqrt{f^2 - n^2}$, and $r = ig = \sqrt{n^2 - f^2}$, then our equation may be written

$$s = e^{-nt}\left[A e^{-rt} + B e^{rt}\right] \qquad (35).$$

If the initial conditions $s = a$ and $s' = 0$ are true for the time $t = 0$, then the constants have the values

$$\left.\begin{aligned} A &= -a(n-r)/2r \\ B &= +a(n+r)/2r \end{aligned}\right\} \qquad (36).$$

and accordingly we have the equation for the instantaneous displacement in the form

$$S=\frac{a}{2r}e^{-nt}\left\{(n+r)e^{rt}-(n-r)e^{-rt}\right\}. \qquad (37).$$

Also the velocity

$$s'=\frac{a}{2r}(r^2-n^2)e^{-nt}(e^{rt}-e^{-rt}), \qquad (38).$$

and the acceleration

$$s''=\frac{a}{2r}(r^2-n^2)e^{-nt}\left[(r-n)e^{rt}+(r+n)e^{-rt}\right] \quad (39).$$

It must now be shown that the motion in question is periodic and for that purpose let us make use of the pair of expressions

$$e^{igt}=\cos gt+i\sin gt,$$
$$e^{-igt}=\cos gt-i\sin gt.$$

The substitution of the trigonometric terms in the above equation gives

$$s=e^{-nt}(A(\cos gt-i\sin gt)+B(\cos gt+i\sin gt))$$
$$=e^{-nt}((A+B)\cos gt-i(A-B)\sin gt)$$

Incorporating the values of the constants A and B gives

$$s=ae^{-nt}(\cos gt+n/g\sin gt). \qquad (40).$$

If n/g be replaced by tan(−α), the last expression becomes

$$s=ae^{-nt}(f/g\sin g(t-\alpha)). \qquad (40').$$

The angular velocity now becomes

$$ds/dt=-ae^{-nt}f(n/g\cdot\sin g(t-\alpha)-\cos g(t-\alpha)) \qquad (41)$$

and the acceleration

$$ds/dt=ae^{-nt}f((n^2-g^2)\sin\cdot g(t-\alpha)). \qquad (42)$$

In the case of Eq.(40) the deflection is zero when $t = 1/g \cdot \tan^{-1}(-g/n)$ and $t = \alpha$ in Eq.(40').

If the body start from rest at one of its elongations, then there results

$$s = a'e^{-nt} \sin\left(gt + \pi/2\right),$$

or

$$s = a'e^{-nt} \cos\left(gt\right). \qquad (43).$$

The velocity is

$$s' = -a'e^{-nt}\left(n\cos \cdot gt - g \cdot \sin \cdot gt\right),$$

$$= -ns + a'e^{-nt} \cdot g \cdot \sin gt. \qquad (44).$$

If the body passes thru the zero position with a velocity v_0. then we may also write

$$s = v_0/g \cdot e^{-nt} \sin gt. \qquad (45)$$

The elongation is attained when $t = 1/g \cdot \tan^{-1}(1/n)$.

and its value is

$$s_{max} = v_0/g e^{-nt} \sin\left(g \cdot 1/g \cdot \tan^{-1} 1/n\right),$$

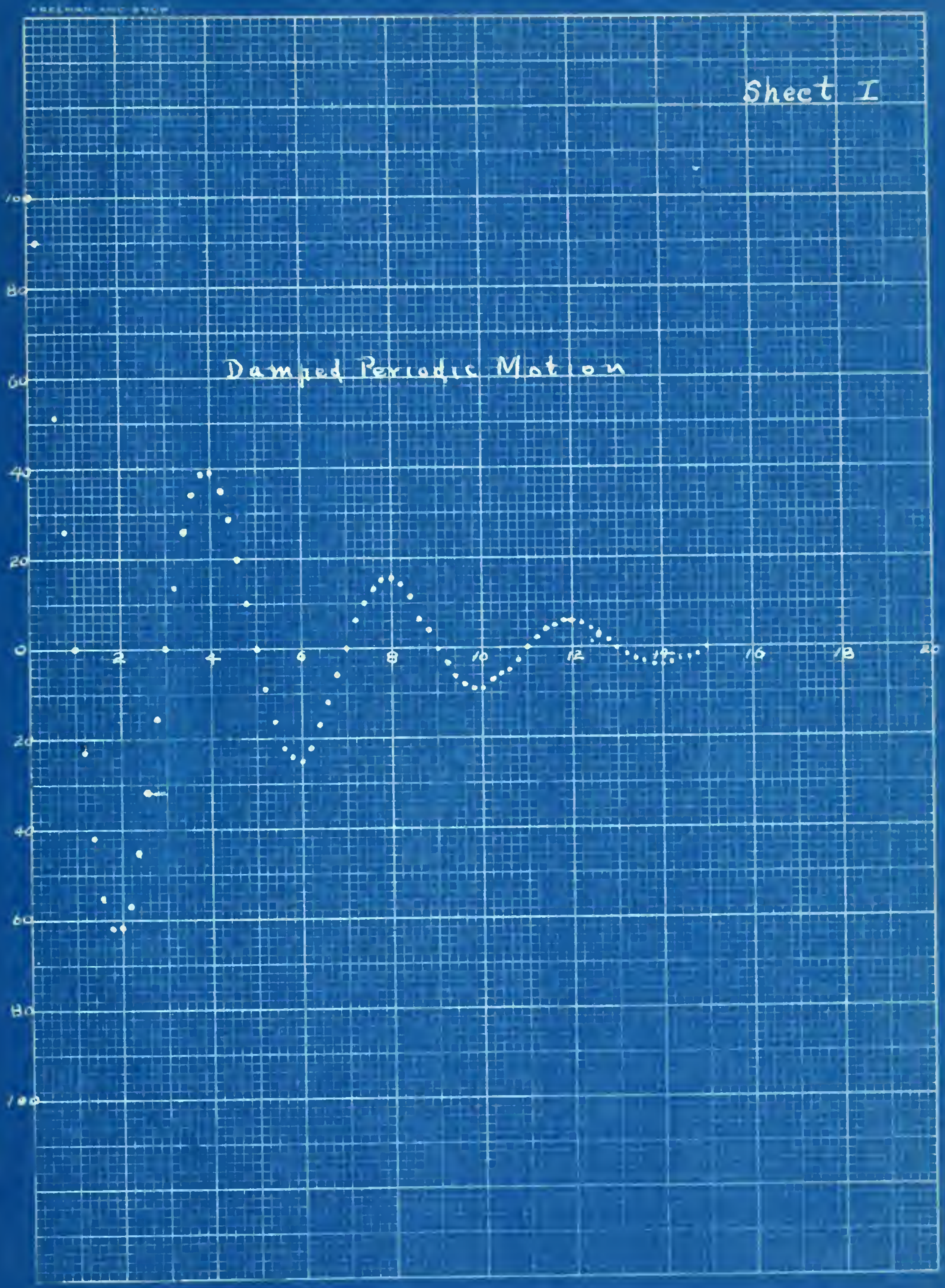
$$= v_0/g e^{-\frac{n}{g}\tan^{-1}1/n} \cdot 1/\left(\sqrt{n^2+1}\right). \qquad (46).$$

Corresponding to the equation

$$s = a'e^{-nt} \sin(gt - \alpha) \qquad (40').$$

it is found that the times of maximum and minimum displacements are given by $t_{max} = \frac{1}{g}\left(\tan^{-1}\frac{g}{n} + \alpha\right) + \left(\frac{2\nu-1}{2}\right)\frac{T_d}{2}$

$t_{min} = \frac{1}{g}\left(\tan^{-1}\frac{g}{n} + \alpha\right)$.



Sheet I

Damped Periodic Motion

q being any whole positive number.

The motion is recognized as being periodic owing to the presence of the trigonometric term. Further, since the elongations on the same side of the origin at equally separated intervals, it follows that the oscillations are isochronous just as in the case of undamped motion. The periodic time is given by $T = 2\pi/g = 2\pi/\sqrt{f^2-n^2}$.

If we accept $s = Ae^{-nt}\sin(gt-\alpha)$ as the (40), form for the general type of damped oscillatory motion, and if the specification of the initial conditions be $S = S_1$, $s' = v_1$, when $t = t_1$ then the determination of the constants, in the most general manner, gives

$$\left.\begin{aligned} \alpha &= gt_1 - \tan^{-1}(s_1 g/v_1), \\ A &= 1/g\sqrt{v_1^2 + 2ns_1v_1 + s_1^2(n^2+g^2)}\ . \end{aligned}\right\} \qquad ([illegible])$$

These two expressions give the values of the constants of integration for any possible combination of initial conditions.

Any constants determined by the aid of the last formulas will be found in correspondence with those which are obtained for the similiar case of undamped motion. If the fist pair of constants be modified by allowing n to become zero, then the second pair of constants is the result. Thus it can be seen that Eq. (40') is closely related to Eq.(11), the two becoming the same when n is zero; i.e. when damping is absent. It is therefore clear that the exponential term arises thru damping alone and from its unchanging negative character, it follows that the displacements or swings suffer diminution in a geometrical ratio. For the same value of t, then, the displacements of the damped and the undamped motions are in the ratio of e^{-nt} to 1.

Chapter III.

We are now prepared to take up the consideration of the interesting case when the coefficient of damping is greater than the coeficient of restitution; i.e. n > f. Returning to Eq.(35) and rewriting as

$$s = \frac{a}{2r}\epsilon^{-nt}\{(n+r)\epsilon^{rt} - (n-r)\epsilon^{-rt}\}, \qquad (48).$$

$$s = \frac{a}{2r}\left\{\frac{n+r}{\epsilon^{(n-r)t}} - \frac{(n-r)}{\epsilon^{(n+r)t}}\right\}, \qquad (48').$$

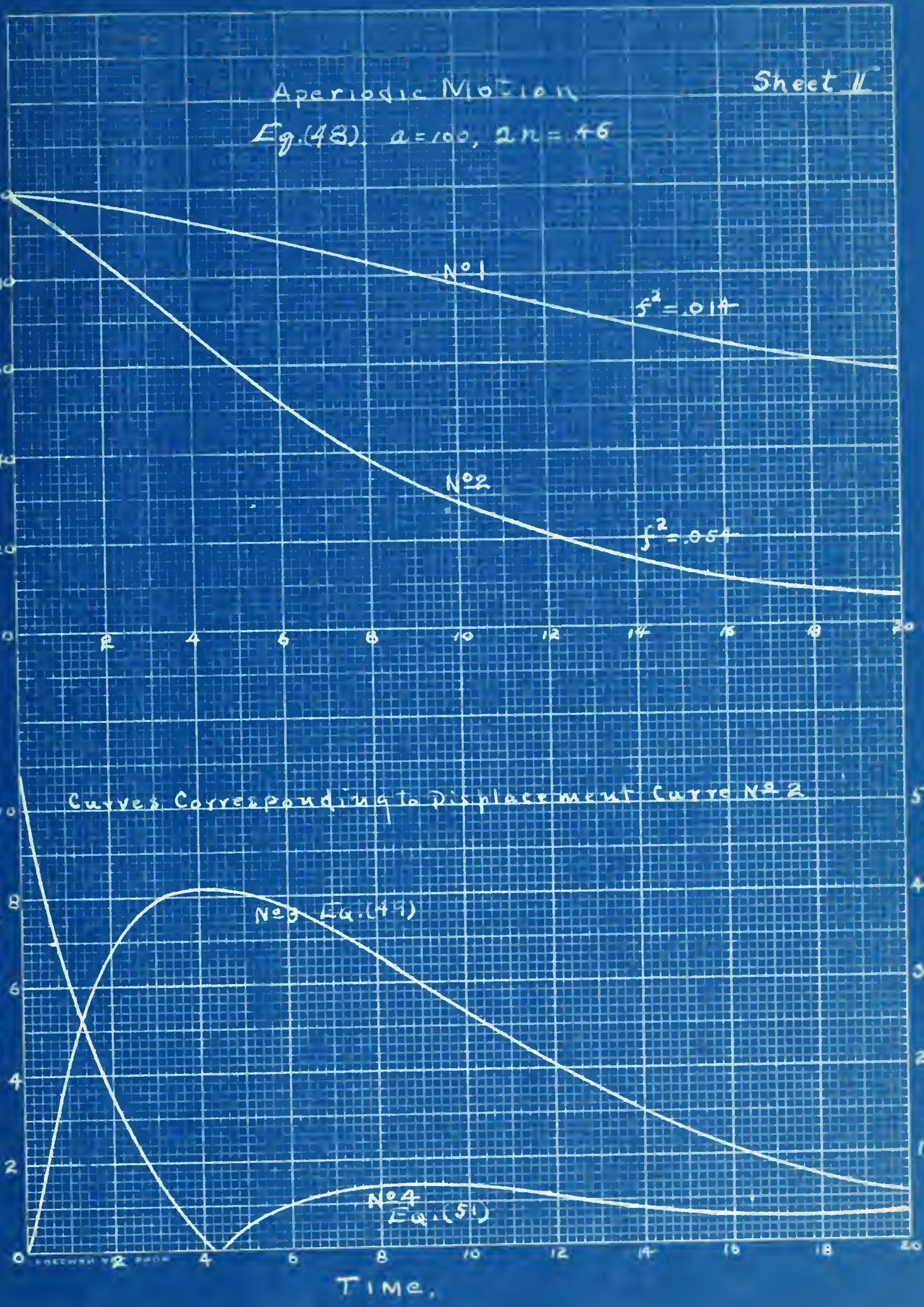
it can be seen, bearing in mind the relationship between n and r, that s suffers a steady diminution from its maximum value s = a, when t = 0, to the value s = 0, when t = ∞; the body therefore gradually comes to rest at the origin and without having passed thru zero during the infinite time that elapses during the motion. This type of motion is called aperiodic, the term originating with du Bois-Reymond.

The velocity, $s' = -\frac{a}{2r}\epsilon^{-nt}f^2(\epsilon^{rt} - \epsilon^{-rt})$, (49).

attains a maximum at $t = \frac{1}{2r}\log\frac{n+r}{n-r}$, (50).

and the acceleration $s'' = \frac{af^2}{2r}\epsilon^{-nt}\left[(n+r)\epsilon^{-rt} - (n-r)\epsilon^{rt}\right]$, (51).

its maximum at $t = \frac{1}{2r}\log\left[(n+r)^2/(n-r)^2\right]$. (52).


Aperiodic Motion
Sheet II
Eq.(48). a=100, 2n=.46
Displacement.
100
80
60
40
20
0
No 1
f²=.014
No 2
f²=.054
2
4
6
8
10
12
14
16
18
20
Curves Corresponding to Displacement Curve No 2
Velocity. No 3.
10
8
6
4
2
0
Acceleration. No 4
5
4
3
2
1
No 3 Eq.(47)
No 4 Eq.(51)
0
2
4
6
8
10
12
14
16
18
20
Time.

To determine the constants to the general equation

$$S = e^{-nt}\left[A e^{-rt} + B e^{rt}\right] \qquad (35).$$

in the most general manner, we easily find them to be

$$\left.\begin{aligned} A &= \frac{\left[S_1(n-r) + v_1\right] e^{nt_1}}{-2r e^{rt_1}}, \\ B &= \frac{\left[s_1(n+r) - v_1\right] e^{nt_1}}{2r e^{rt_1}}. \end{aligned}\right\} \qquad (53).$$

If time is to be measured from the instant of transit thru zero, then by the formulas just derived we have $A = -v_o/2r$, $B = v_o/2r$. The equation of motion then becomes

$$S = \frac{v_0}{2r} e^{-nt}\left[e^{rt} - e^{-rt}\right]. \qquad (54).$$

The corresponding expression for the velocity is

$$\frac{ds}{dt} = \frac{v_0}{2r} e^{-nt}\left[e^{rt}(r-n) + e^{-rt}(r+n)\right], \qquad (55).$$

and for the acceleration

$$\frac{d^2s}{dt^2} = \frac{v_0}{2r} e^{-nt}\left[e^{rt}(n-r)^2 - e^{-rt}(n+r)^2\right]. \qquad (56).$$

As the body moves away from zero with an initial velocity v_o, it will attain a maximum position s_m at a time t_m, say, and whose value is, as we have already seen, $t_m = \frac{1}{2r} \log \frac{n+r}{n-r}$. (57). The magnitude of the swing is given by

$$S_m = \frac{v_0}{2r}\left(\frac{n+r}{n-r}\right)^{-\frac{n}{2r}}\left[\left(\frac{n+r}{n-r}\right)^{1/2} - \left(\frac{n+r}{(n-r}\right)^{-\frac{1}{2}}\right]. \quad (58$$

The position of the body at any time after elongation and measured from that instant, as indicated by the relation $t_1 = t - t_m$, is given by, as can be seen from the nature of the case,

$$S = \frac{S_m}{2r} e^{-nt_1}\left\{(n+r)\epsilon^{rt_1} - (n-r)\epsilon^{-rt_1}\right\}. \qquad (58)'$$

And finally when damping is very great, arising from the relatively large value of n, then n and r do not differ materially, and the last equation becomes

$$S = S_m \epsilon^{(r-n)t_1}. \qquad (59)..$$

The ratio of the deflection s as given by Eq. (37′) to the first swing s_m is

$$2r\frac{S}{S_m}\epsilon^{n(t-t_m)} = \left\{(n+r)\epsilon^{r(t-t_m)} - (n-r)\epsilon^{-r(t-t_m)}\right\} (60).$$

This implicit relation between the fraction s/s_m and the time $(t - t_m)$, the interval that has elapsed since the elongation, can be solved by cut-and-try methods for either the s/s_m fraction or the time interval when the other is known or assumed.

If the coefficient of damping is so large in comparison to the coefficient of restoration that the latter is negligible, then $n = r$ and Eq.(31) is, in its general form, $S = Ae^{-2nt} + B$, (61).

and for the special conditions $s = s_1$, $v = v_1$, $t = t_1$ then $A = -v_1 e^{2nt_1}/2n$; $B = s_1 + v_1/2n$. When these special relations refer to the circumstance of a the body starting from rest at its elongation then the equation of motion becomes $S = a$, since $A = 0$ and $B = a$. That is to say the body does not move from the initial position and there is, therefore, no motion. The motion is completely damped out of existence. The graph of this motion is then clearly a straight line parallel to the time axis and passing thru the point $s = a$ on the displacement axis. Further it follows that this line is the asymptote of all the less highly damped curves having the same starting point $s = a$.

As a closing paragraph to the chapter, it will be interesting to assume that the heavily damped body is given a blow, imparting to it a velocity ± c, at the begining of time. Accordingly, the constants are

$$A = \pm c/2n, \qquad B = a \pm c/2n$$

and the motion is represented by

$$s = a \pm c/2n\,(1-e^{-2nt}). \qquad (62).$$

the velocity by

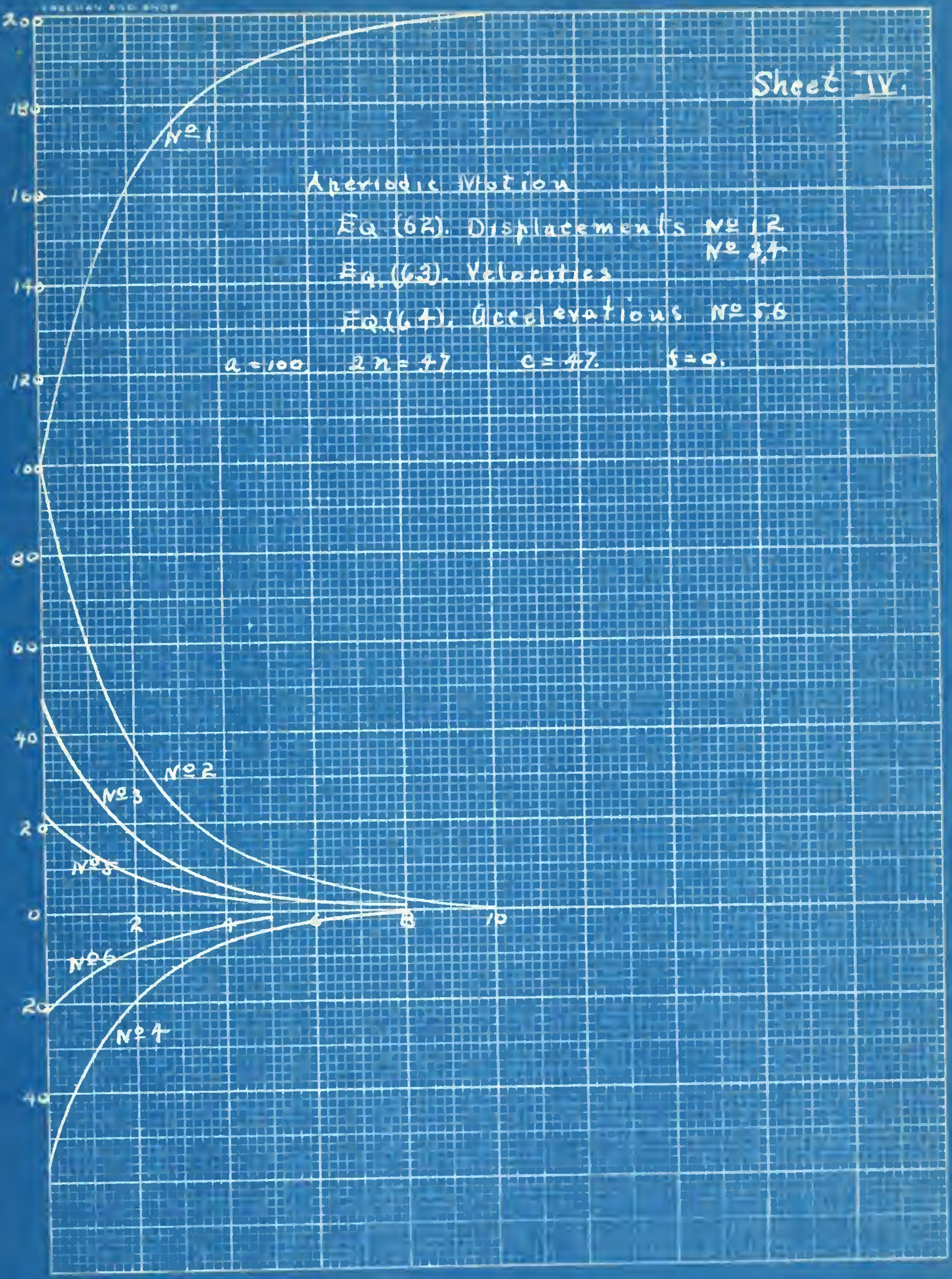
$$s' = \pm ce^{-2nt}, \qquad (63).$$

and the acceleration by

$$s'' = \pm 2cne^{-2nt}. \qquad (64).$$

The final position of rest is given by

$$s = a \pm c/2n. \qquad (65).$$


Sheet IV.
Aperiodic Motion
Eq. (62). Displacements № 1,2
№ 3,4
Eq. (63). Velocities
Eq. (64). Accelerations № 5,6
a = 100
2n = 47
c = 47.
δ = 0.
№ 1
№ 2
№ 3
№ 4
№ 5
№ 6
200
180
160
140
120
100
80
60
40
20
0
20
40
2
4
6
8
10

Chapter IV.

The fundamental condition underlying the present case is defined by the relation $N = f$ and it constitutes the transition from the first to the second of the preceeding cases. The coefficient of damping is now equal to the coefficient of restitution. The radical r vanishes, $m_1 = m_2$ and we have for investigation the very simple equation

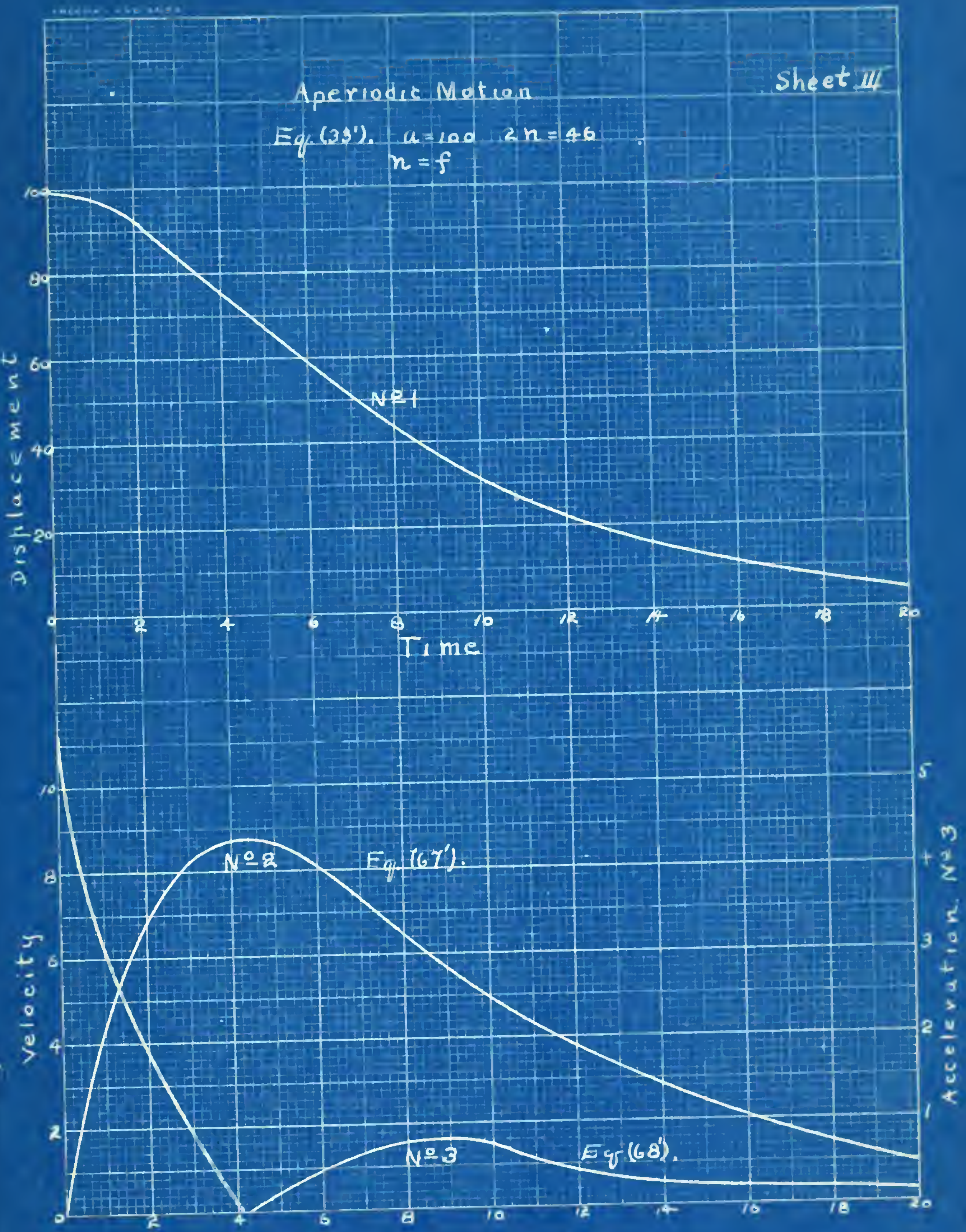
$$s = (A + Bt)e^{-nt}. \qquad (33).$$

The constants, expressed in their most general form, are $A = e^{nt_1}(s_1 - (v_1 + ns)t_1)$, $B = e^{nt_1}(v_1 + ns_1)$. (66).

If the motion and time commence simultaneously from the position of maximum displacement, then the above equation becomes $s = ae^{-nt}(1 + nt)$. (33')

The velocity, $s' = -an^2te^{-nt}$, attains a maximum at the (67') time $t = 1/n$ and a value of $s' = -an/e$. (66')

The acceleration $s'' = an^2e^{-nt}(nt - 1)$ attains a maximum (68') of $s'' = an^2/e^2$ at $t = 2/n$. (67)


Sheet III
Aperiodic Motion
Eq. (33'). a = 100 2n = 46
n = f
Displacement
100
80
60
40
20
0
No 1
2
4
6
8
10
12
14
16
18
20
Time
Velocity
10
8
6
4
2
0
No 2
Eq. (67').
No 3
Eq (68').
Acceleration. No 3
5
4
3
2
1
2
4
6
8
10
12
14
16
18
20

The damping in this case is said to be critical and for reasons that will appear later. On that account it will be desirable to investigate the case along a line that agrees with it's practical application. If the initial conditions be specified as $s = 0$, $s' = v_0$ and $t = 0$, that is to say, the body at zero receives an impulse and starts off with a velocity v_0 at the begining of time. Our equation is

$$s = v_0 e^{-nt} t.$$

If $n > f$ the motion is more aperiodic, and if $n < f$ oscillations take place. Accordingly, the present relation, $n = f$, is the one that must be possessed by the moving system if it is to return to zero in the shortest time and without executing vibrations. It is, then, the condition of minimum aperiodicity and the one to be met in the oscillograph.

Let the time of the first swing be t_1, its value be s_1, and let the period and swing of the motion.

if there no damping, be T_0 and S_0. Now $t_1 = 1/n$ and $s_1 = v_0/ne$ and from the first chapter $S_0 = v_0 T_0/2\pi$, $T_0 = 2\pi/f = 2\pi/n$. Hence $s_1 = v_0/fe$, $S = v_0/f$, $s_1 = S_0/e$, $s_1 = v_0 T_0/2\pi e = v_0 T_0/17.080$ and $t_1 = T_0/2\pi$.

Thus it is seen that there is a very simple relationship between the time of the first swing of the damped motion and the period of the same motion if undamped, and that this first-named time is the square-root of the coefficient of damping.

Now Eq.(33′) gives the position of the system at any time after the elongation, occuring at t_1, and accordingly the value of the displacement at the time $t + t_1$ is given by $S = s_1 e^{-nt_1}(1 + n(t_1 + t))$. (68).

A few computations disclose the fact that when the time elapsed since elongation, t, bears the following ratios to t_1, the time of elongation, then the value of the displacement expressed in terms of the maximum

displacement, s_1, are given in the table.

t/t_1	2.1	2.9	3.7
s/s_1	1/100	1/1000	1/10,000

From these values a general idea may be obtained of the quickness of the return of the system to zero, the operation being completed in about three times the period of the first swing.

Chapter V.

Concerning Damping Factors.

If in Eq.(40') s_1 and s_3 are used to designate the maximum values of two consecutive swings on the same side of the origin, and if they occur at times t_1 and t_3 we have

$$s_1 = a\,e^{-nt_1}(f/q \sin(qt_1-\alpha)),$$

$$s_3 = a\,e^{-nt_3}(f/q \sin(gt_3-\alpha)),$$

and also

$$s_1/s_3 = e^{nt_3}/e^{nt_1}.$$

If the periodic time of the damped motion be T, and as $t_3 = t_1 + T$, we have

$$e^{nT} = s_1/s_3 = s_3/s_5 = s_5/s_7 = \cdots$$

$$= s_2/s_4 = s_4/s_6 = s_6/s_8 = s_n/s_{n+2} = k^2, \text{say.}$$

Therefore

$$s_1/s_5 = e^{2nT} = k^4, \qquad (69).$$

$$s_p/s_m = e^{\left(\frac{m-p}{2}\right)nT} = (k^2)^{\left(\frac{m-p}{2}\right)} \text{ and}$$

$$s_1/s_2 = k = e^{nT/2}.$$

And finally $\log(s_1/s_2) = 1/2nT = \log k = \lambda$, say. (70).

Now the log of the constant relation between the successive swings, and represented by λ, is called the logarithmic decrement and is the natural logarithm of the damping constant, k.

From $\lambda = \log(s_1/s_2) = \log(s_2/s_3) = \ldots \log(s_m/s_{m+1})$, we have $2\lambda = \log(s_1/s_3)$; $3\lambda = \log(s_1/s_4)$ and $m\lambda = \log(s_1/s_{m+1})$, or finally,

$$\lambda = (1/m)\cdot\log(s_p/s_{p+m}). \qquad (70')$$

Hence the constant may be readily obtained by taking the naturallogarithm of the ratio of the pth and the (m+p)th swings as indicated by the last expression.

From the preceeding fundamental relations we shall now derive the approximate formula for the damping factor as commonly given and used. As k is defined by the ratio s_n/s_{n+1}, it follows that the damping constant is the multiplying factor by which any swing, n, is to be obtained from the (n+1). The swings diminish, therefore, in geometrical progression and as damping is present in each swing, k is the product of the damping effect in the n and the (n+1) swing. That is to say, the damping effect present in a single swing is $\sqrt{k}$ and hence if S_0 be the undamped and s the

damped value of an oscillation. then $S_0/s = \sqrt{k}$.

or $S_0/s = e^{\lambda/2}$, as $\log k = \lambda$, $e^{\lambda} = k$. Accordingly,

$$\log S_0 = \lambda/2 + \log s$$

$$S_0 = e^{\lambda/2 + \log s}$$

$$= e^{\lambda/2} \cdot e^{\log s}$$

$$= e^{\lambda/2} s$$

$$= s\left(1 + \lambda/2 + \lambda^2/1\cdot 2 + \lambda^3/1\cdot 2\cdot 3 + \cdots\right) \quad (71).$$

which is the ordinary formula.

We shall now take up the deduction of the true expression for the damping factor. If in Eq.(40')

$S = Ae^{-nt}\sin(gt - \alpha)$ subject to the initial conditions

$s = 0$, $s' = v_1$, $t = 0$, then $A = v_1/g$ and $\alpha = 0$; and as $g = 2\pi/T$

we have $S = \frac{v_1}{g} e^{-\frac{2\lambda t}{T}} \sin\left(\frac{2\pi t}{T}\right)$,

and $S' = -\frac{v_1}{g} \cdot \frac{2\lambda}{T} e^{-\frac{2\lambda t}{T}} \sin\left(\frac{2\pi t}{T}\right) + \frac{v_1}{g} e^{-\frac{2\lambda t}{T}} \cdot \frac{2\pi}{T} \cos \frac{2\pi t}{T}$.

The first elongation occurs when $t_1 = \frac{T}{2\pi} \tan^{-1} \frac{\pi}{\lambda}$.

Now since $\sin\left(\frac{2\pi t_1}{T}\right) = \frac{\pi}{\sqrt{\pi^2 + \lambda^2}}$ and $K = e^{\lambda} = e^{\frac{nT}{2}}$

or $e^{-nt_1} = K^{-\frac{2t_1}{T}}$ we have as the value of the

displacement at t_1

$$S_1 = \frac{v_1}{g} K^{-\frac{2t_1}{T}} \frac{\pi}{\sqrt{\pi^2 + \lambda^2}} = \frac{v_1 T}{2\pi} K^{-\frac{1}{\pi}\tan^{-1}\pi/\lambda} \cdot \frac{\pi}{\sqrt{\pi^2 + \lambda^2}}. \quad (72).$$

This then is the expression for the displacement in damped periodic motion and given in terms of the logarithmic decrement, λ, the initial velocity, and the periodic time T.

It is interesting, tho for our purpose it is of no importance. to note that the velocities possessed by the system at the times 0, T/2, T, 3/2T and obtained from Eq.(72) are v_1, $v_1 e^{\lambda}$, $v_1 e^{2\lambda}$; that is.

$$v_1/v_2 = e^{\lambda} = v_2/v_3 = k.$$

Or we see that the same damping action is present in the velocities at the time of passage thru zero as is present in the swings themselves.

If damping were absent and the same initial velocity were had, then we would find

$$S_0 = T_0 v_1/2\pi$$

and the value of the damped swing may now be given as

$$S_1 = \frac{S_0 T}{T_0} K^{-\frac{1}{\pi}\tan^{-1}\frac{\pi}{\lambda}} \cdot \frac{\pi}{\sqrt{\pi^2+\lambda^2}} \quad (73).$$

Since $T = \frac{T_0}{\pi}\sqrt{\pi^2+\lambda^2}$, the last equation becomes

$$S_0 = s_1 k^{\frac{1}{\pi}\tan^{-1}\frac{\pi}{\lambda}}; \qquad (74).$$

$$= s_1 e^{\lambda/\pi \tan^{-1}\pi/\lambda}, \qquad (75).$$

and similiarly $v_1 = \frac{\pi S_1}{T_0} K^{1/\pi \tan^{-1}\pi/\lambda}$. (76).

When λ becomes infinite, that is to say, when the condition is one of critical damping.

$$S_0 = s_1 e$$

$$= 2.7183 \cdot s_1 \qquad (77).$$

If the right-hand side of Eq.(75) be expanded as a series we have

$$S_0 = s_1\left[\left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{8} + \cdots\right) - \left(\frac{\lambda^2}{\pi^2} + \frac{\lambda^3}{2\pi^2} + \frac{\lambda^4}{8\pi^2} + \cdots\right)\right]; \qquad (78).$$

a series, convergent, if $\lambda < \pi$.

It can be easily shown that the value of the true damping factor $e^{\lambda/\pi \tan^{-1}\pi/\lambda}$ becomes equal to the approximate value $e^{\lambda/2}$ for the value of $\lambda = 0$ only.

It is interesting to note in passing that

$$2n = \lambda / T \qquad (79).$$

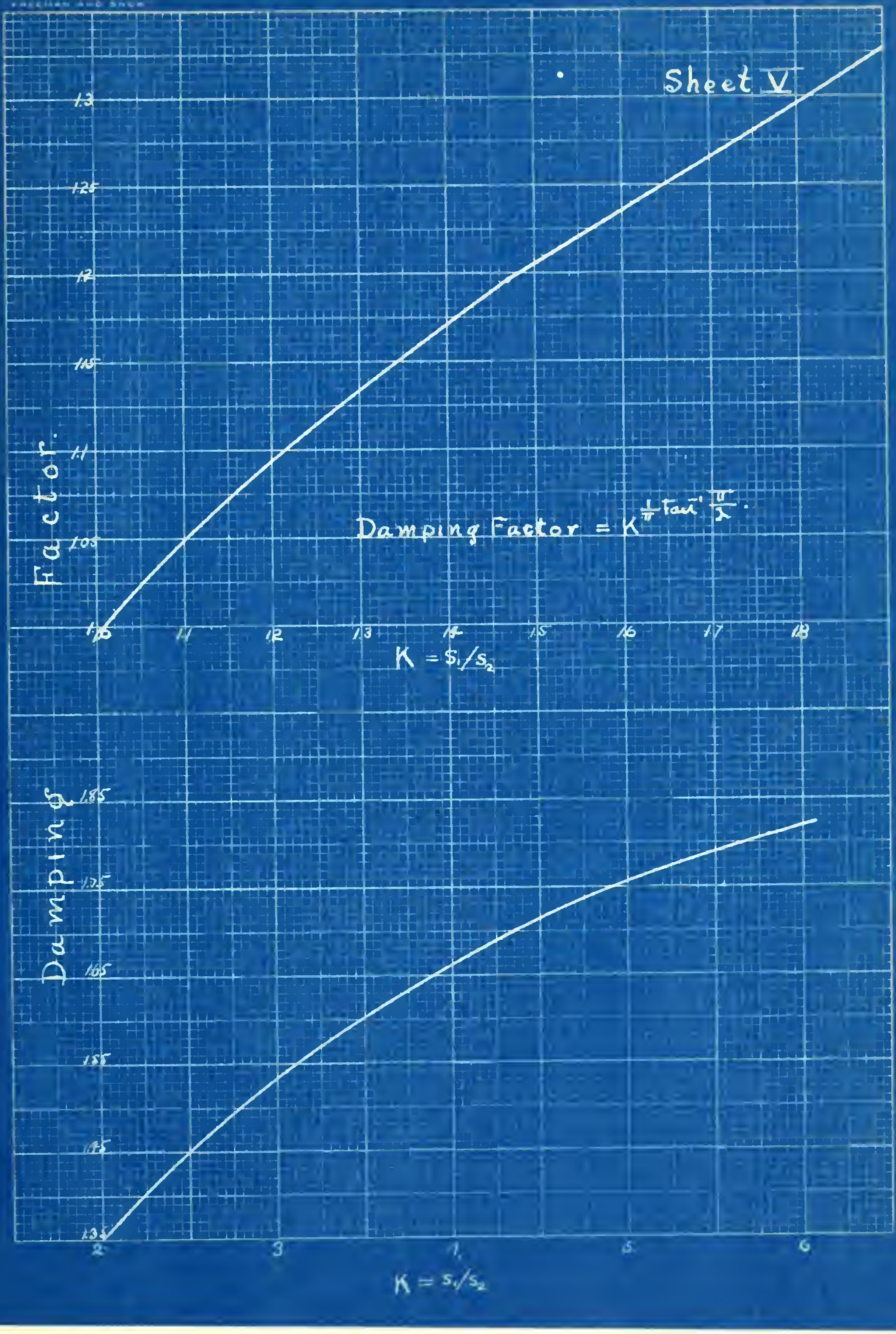
a relation that enables the constant n to be determined.

And from the second equation

$$f = 2\pi / T_0 \qquad (80)$$

the other equational constant may be found.

Sheet V


Damping Factor.
Damping Factor = $K^{\frac{1}{\pi}\tan^{-1}\frac{\pi}{\lambda}}$.
1.3
1.25
1.2
1.15
1.1
1.05
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
K = s_1/s_2
1.85
1.75
1.65
1.55
1.45
1.35
2
3
4
5
6
K = s_1/s_2

Now that we have derived the true expression for the damping factor, it will be desirable to make a comparison of the various forms in which this quantity is encountered and used, with the correct one.

Attention will first be paid to the form that was more recently proposed by H.H. Smith but whose origin is to be placed to the credit of Prof. W. Stroud. Its relation to that which has gone before may be easily shown in the following way. Now, clearly, we have $\tan^{-1}\frac{\pi}{\lambda} = \frac{\pi}{2} - h$ (81). where h is a constant. Hence $\tan^{-1}\pi/\lambda \Big/ \pi/\lambda = \lambda/2 - \frac{h\lambda}{\pi}$ and as the true relation between the damped and the undamped deflections is given by $S_0 = S_1 e^{\lambda/\pi \tan^{-1}\pi/\lambda}$, we may write $S_0' = S_1 e^{\lambda/2} \Big/ e^{\frac{h\lambda}{\pi}}$ (82). If in Eq.(81) λ is negligible compared to π, then h is negligible in comparison to $\pi/2$ and therefore $\frac{h\lambda}{\pi}$ is negligible with respect to unity. Accordingly, Eq.(82) is sensibly equal to $S_0' = S_1 e^{\lambda/2}$. (83).

But by definition, $s_1/s_3 = e^{2\lambda}$ and hence we have as Stroud's form for the damping factor

$$s_0' = s_1\left(\frac{s_1}{s_3}\right)^{1/4}, \qquad (84).$$

an expression that can only be regarded as representing the case when λ is small in comparison with π.

An examination of Eq.(84) and Eq.(82) shows that the value of the undamped swing as given by the former is always larger than that calculated by the aid of the latter. The error introduced by the use of the Stroud-Smith formula increases with the logarithmic decrement and the ratio of the two values is

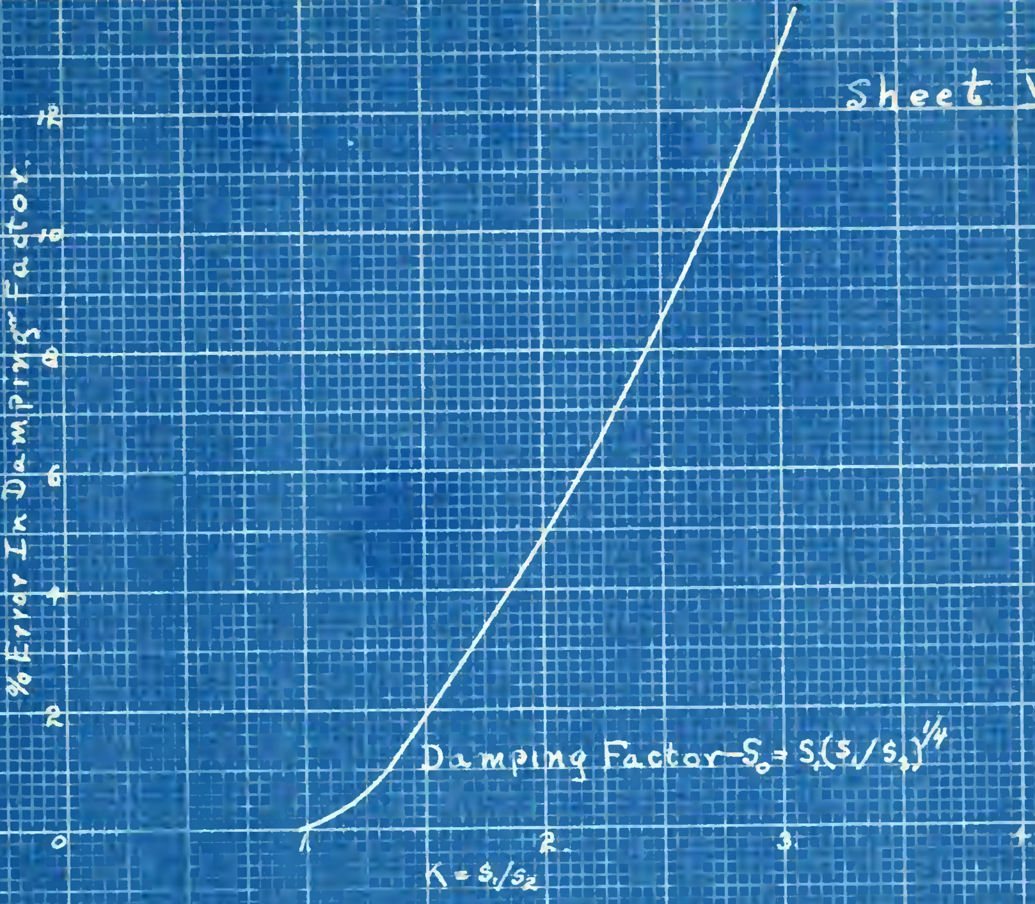
$$\frac{s_0'}{s_0} = \frac{s_1 e^{\lambda/2} e^{\frac{h\lambda}{\pi}}}{s_1 e^{\lambda/2}} = e^{h\lambda/\pi} = e^{\left(\lambda/2 - \lambda/\pi \tan^{-1} \pi/\lambda\right)}. \qquad (85).$$

Bearing in mind that h is a directly varying function of λ, we see that the two swings become the same only when λ is zero.

The per cent error in any case is given by

$$\begin{aligned}\frac{S_0' - S_0}{S_0}100 &= \left(e^{h\lambda/\pi} - 1\right)100 \\ &= \left[\frac{\sqrt{K}}{K^{1/\pi \tan^{-1} \pi/\lambda}} - 1\right]100. \qquad (86).\end{aligned}$$

Sheet VI


% Error In Damping Factor.
12
10
8
6
4
2
0
1
2
3
4
Damping Factor—$S_0 = S_1(S_1/S_2)^{1/4}$
$K = S_1/S_2$

The graph of this last equation enables the error present in the use of this simple form to be obtained by inspection. This equation, plotted between k and per cent error, is given on Curve-sheet No. VI.

Lastly, it may be seen that $e^{-\frac{h\lambda}{\pi}}$ as shown in Eq.(85), is the correction-factor to the correction-factor. In critical damping, when $\lambda=\infty$, the error becomes infinite.

We shall now pass on to the consideration of the standard formula $S_0' = S_1(1+\lambda/2)$ (87) which consists of the first two terms of the expansion of $e^{\lambda/2}$, as was shown in the earlier part of the chapter.

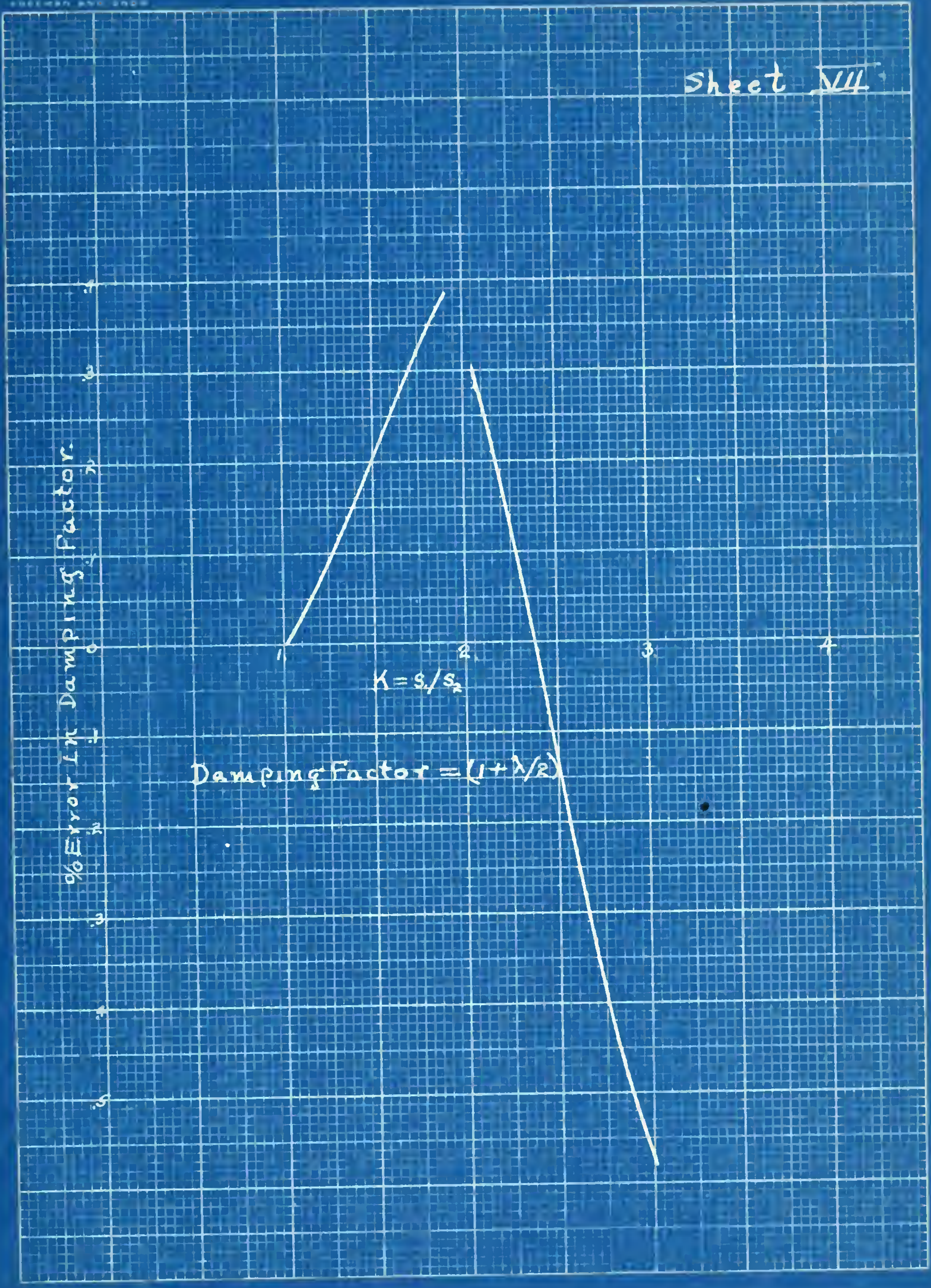
The error present in this case is clearly given by

$$\frac{S_0'-S_0}{S_0}100 = \left(1 - \frac{1+\lambda/2}{K^{\frac{1}{\pi}\tan^{-1}\frac{\pi}{K}}}\right)100 \qquad (87)'.$$

This equation is plotted on Curve-sheet No. VII.

Here, as before, the error becomes infinite with

Sheet VII


% Error in Damping Factor.
4
3
2
1
0
1
2
3
4
5
1
2
3
4
K = S₁/S₂
Damping Factor = (1 + λ/2)

critical damping. But we know that $S_0 = S_1 e = 2.718 s_1$ in this instance.

From the graph of this percent error equation it is seen that the magnitude of the error is small for such values of k as are ordinarliy encountered, and accordingly this form of damping factor can be used in the average case.

We have now to consider the empirical formula

$$S_0' = S_1 + \frac{1}{4}\left(\frac{S_1 - S_3}{1}\right) \qquad (88)$$

Since $s_1/s_3 = e^{2\lambda} = (1 + 2\lambda + ---- \qquad)$

$= 1 + 2\lambda$, if higher powers be neglected.

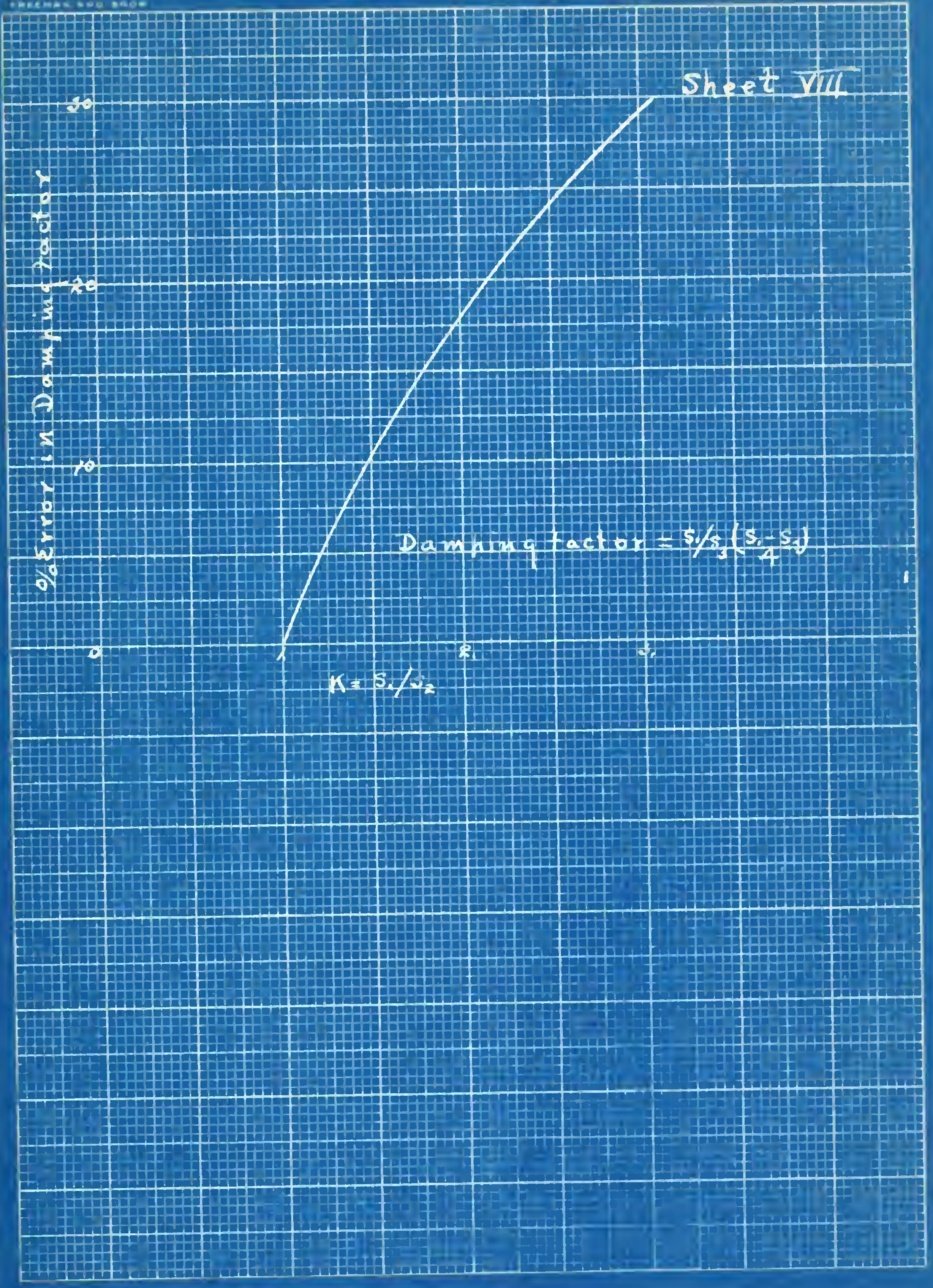
we have $(\lambda/2) = \frac{S_1 - S_3}{4S_3}$ and from Eq.(87) $S_0' = S_1 + S_1\left(\frac{S_1 - S_3}{4S_3}\right)$,

If λ be small, then s_1/s_3 is comparable with unity and there results $S_0' = S_1 + \frac{1}{4}(S_1 - S_3)$. Q.E.D.

The per cent error is given by

$$100\frac{S_0' - S_0}{S_0} = \left(1 - \frac{5K^{-\frac{1}{\pi}\tan^{-1}\frac{\pi}{\lambda}} - K^{-\frac{3\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}}}{4}\right)100. \qquad (89).$$

and an inspection of its graph on Curve-sheet No. $\underline{\overline{VIII}}$ shows that the error involved in its use is much larger than that incurred thru the application of


Sheet VIII
30
20
10
0
% Error in Damping factor
Damping factor = $S_1/S_3\left(\frac{S_1-S_2}{4}\right)$
.8
.9
1
$K = S_1/S_2$

either of the preceeding simplified forms.

Finally, mention may be made of the form

$S_0' = S_1 e \lambda/\pi = .865\lambda S_1$ and proposed for use when

is very large. The application of this makeshift could not be countenanced in any except the most unprecise work.

On pages 43, 44 there are arranged in tabular form the numerical values of correction factors for damping as calculated by the several preceeding formulas.

Correction Factors.

		True	Common	Empirical	Stroud
K	λ	$K^{\frac{1}{\pi}\tan^{-1}\frac{\pi}{\lambda}}$	$1+\lambda/2$	$1+\frac{1-K^{-2}}{4}$	$K^{1/2}$
1.	0.	1.	1.	1.	1.
1.01			1.049	1.005	1.005
1.021			1.010	1.010	1.010
1.023	.023	1.012			
1.032			1.015	1.015	1.015
1.043			1.021	1.020	1.021
1.047	.0461	1.023			
1.055			1.026	1.025	1.026
1.066			1.029	1.030	1.032
1.072	.0691	1.035			
1.084			1.038	1.035	1.038
1.091			1.044	1.040	1.044
1.096	.0921	1.046			
1.105			1.050	1.045	1.050
1.118			1.066	1.050	1.057
1.122	.1151	1.058			
1.148	.1382	1.079			
1.175	.1612	1.08			
1.196			1.089	1.075	1.093
1.202	.1842	1.093			
1.230	.2072	1.104			
1.259	.2303	1.116			
1.288	.2533	1.128			
1.291	.2		1.128	1.100	1.136
1.318	.2765	1.139			
1.349	.2993	1.151			
1.380	.3224	1.163			
1.413	.3454	1.174			
1.414			1.173	1.125	1.189
1.445	.3684	1.184			
1.479	.3914	1.193			
1.514	.4145	1.209			
1.549	.4375	1.221			
1.585	.4605	1.232			
1.622	.4835	1.244			
1.660	.5066	1.256			
1.682			1.229	1.150	1.257
1.698	.5296	1.267			
1.738	.5526	1.279			
1.770			1.301	1.175	1.351
1.779	.5756	1.290			
1.820	.5987	1.301			
1.905	.6447	1.324			
1.950	.6677	1.336			
1.999	.6908	1.347			
2.042	.7138	1.358			
2.089	.7368	1.369			
2.138	.7599	1.381			

K	λ	$K^{\frac{1}{\pi}\tan^{-1}\pi/\lambda}$	$1+\lambda/2$	$1+\frac{1-K^{-2}}{4}$	$K^{1/2}$
2.183	.7829	1.392			
2.239	.8059	1.403			
2.240			1.4023	1.200	1.495
2.291	.8289	1.414			
2.344	.8520	1.425			
2.399	.8750	1.436			
2.455	.8980	1.446			
2.512	.9210	1.458			
2.570	.9441	1.469			
2.630	.9671	1.479			
2.692	.9901	1.490			
2.754	1.013	1.501			
3.020	1.105	1.542			
3.13			1.576	1.225	1.778
3.802	1.336	1.644			
5.012	1.612	1.755			
6.026	1.796	1.824			
7.943	2.72	1.919			
9.550	2.256	1.967			
10.00	2.302	1.989			
10.00			2.151	1.248	3.162
∞	∞	2.718	2.718	1.250	∞

Chapter VI.

We now pass on to the consideration of aperiodic motion when the body has an initial velocity at the begining of time. This part of our subject is necessarily closely related to that which has preceeded and while it involves not only much that is of general interest it suppliesthe concluding chapters that make the discussion complete.

Bearing in mind that the fundamental equations underlying aperiodic motion are $S=e^{-ut}(Ae^{-rt}+Ce^{rt})$, (35). when the coefficient of damping is greater than the coefficient of restitution, and $S=e^{-ut}(A+Bt)$, (33). when the two coefficients are equal, we have two conditions to deal with and which will be designated as the first and second cases respectively.

Let us assume that the initial velocity is —c. which will give it a direction the same as that of

the restoring force, at $t = 0$. Then, as $t = 0$, $s = a$, and $s' = -c$, we obtain as the values of the constants of integration, A, B, in equations (53), (53'), and Eq. (35) becomes, in case I, $A = \frac{c - a(n-r)}{2r}$, $B = \frac{-c + a(n+r)}{2r}$.

$$S = e^{-nt}/2r\left[\{c - a(n-r)\}e^{-rt} - \{c - a(n+r)\}e^{rt}\right] \qquad (90)$$

In case II, the constants are found to be and Eq. (33)

$$A = a \quad , \quad B = -c + na.$$

becomes

$$S = e^{-nt}(a - (c - na)t). \qquad (90')$$

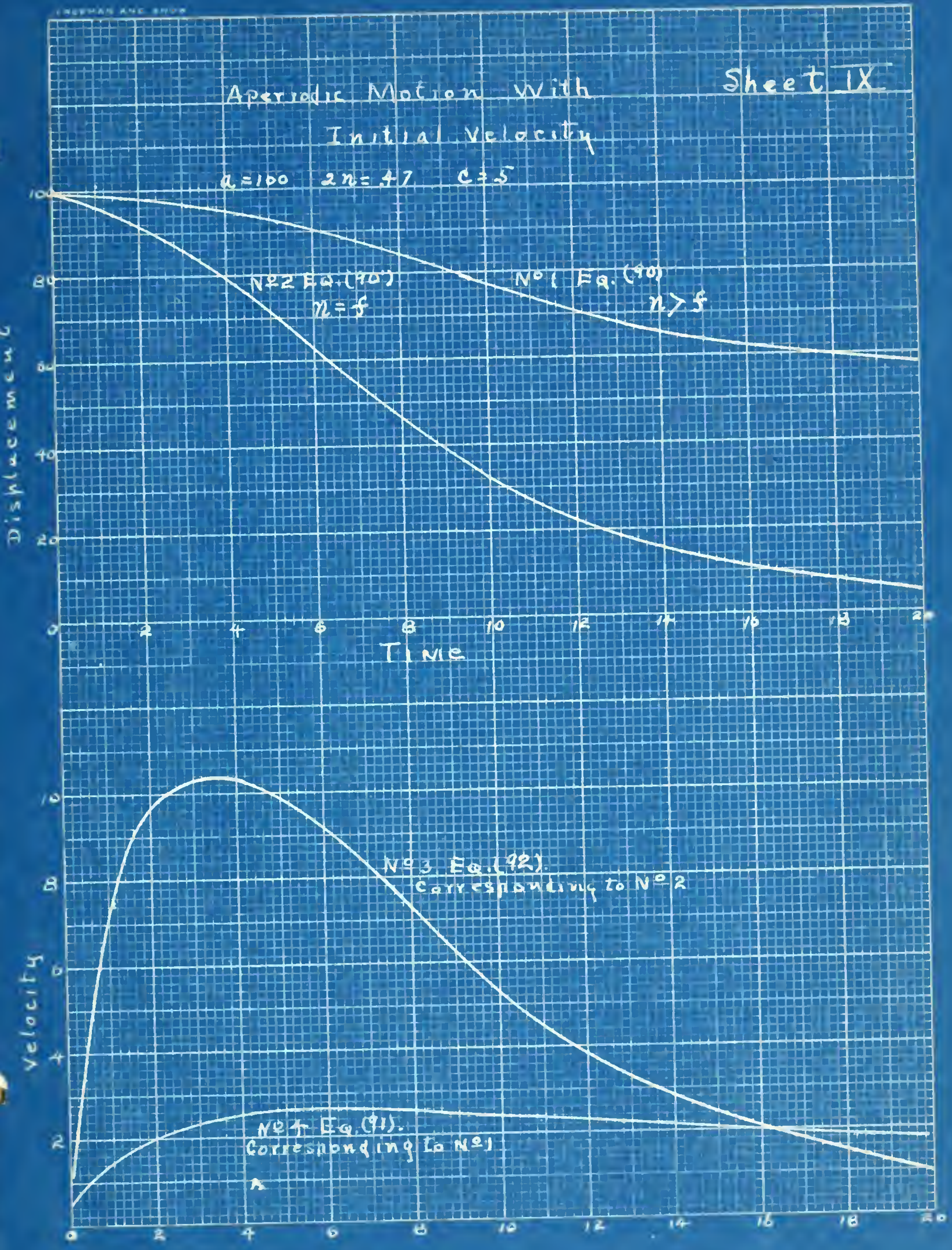
The motion is still aperiodic as may be seen by noting that $s = 0$ when $t = \infty$; tho, as we shall see, a value of $-c$ greater than a certain critical magnitude causes the body to pass thru zero, attain a maximum position on the other side and then creep back to the zero position, under no circumstances can oscillations take place.

The velocitied are easily found to be

$$s' = -e^{-nt}\left\{n\left[\frac{c - a(n-r)}{2r}e^{-rt} - \frac{c - a(n+r)}{2r}e^{rt}\right]\right\}, \qquad (91).$$

and

$$s' = e^{-nt}\left[-n(na - c)t - c\right]. \qquad (92).$$



Sheet IX

Aperiodic Motion with Initial Velocity

$a = 100 \quad 2n = .47 \quad c = .5$

and the accelerations

$$s'' = (n^2+r^2)s + 2nre^{-nt}e^{rt}\left(c - \frac{a(n+r)}{2r}\right), \qquad (93).$$

and $$s'' = e^{-nt}\left[-n(a-(c-na)) - ns'\right\}. \qquad (94).$$

If the left-hand member of Eqs.(90) and (90')

be placed equal to zero and the corresponding values

of t found, we have $t_{s=0} = \frac{1}{2r}\log\frac{c-a(n-r)}{c-a(n+r}$, (95),

and $t_{s=0} = \frac{a}{c-na}$, that is to say, (96).

these are the times of transit of the body thru the origin when the initial velocity is $-c$.

In the same manner, the velocities specified in Eqs. (91) and (92) may be shown to have a zero value at times given by $t_{v=0} = \frac{1}{2r}\log\frac{(n+r)\{c-a(n-r)\}}{(n-r)\{c-a(n+r)\}}$ (95').

and $t_{v=0} = \frac{c}{n(c-na)}$. (96').

Their maximum values occur when

$$t_{v=max.} = \frac{1}{2r}\log\frac{(n+r)^2\{c-a(n-r)\}}{(n-r)^2\{c-a(n+r)\}} \qquad (95'').$$

and $$t_{v=max} = \frac{2c-an}{n(c-an)}. \qquad (96'').$$

If the two curves represented by Eqs.(90),(90') be examined for points of inflection, points at which the curvature changes but which are neither maxima nor

mimima, we find that the deflection curves have such points at $t_{spi} = \frac{1}{2r} \log \frac{(n+r)^2 \{c - a(n-r)\}}{(n-r)^2 \{c - a(n+r)\}}$, (97).

and $$t_{spi} = \frac{2c - na}{n(c - na)};$$ (97').

and the velocity curves at
$$t_{vpi} = \frac{1}{2r} \log \frac{(n+r)^3 \{c - a(n-r)\}}{(n-r)^3 \{c - a(n+r)\}},$$ (98).

and $$t_{vpi} = \frac{3c - 2an}{5(c - an)}.$$ (98').

An inspection of the values of $t_{s=0}$, $t_{v=max}$) t_{spi} and t_{vpi} for each of the two cases, shows that we have an arithmetical series. In the first case, the constant difference is $\frac{1}{2r} \log \frac{n+r}{n-r}$, (99).

and in the second $1/n$. (100).

In the various values of the time given above, it is seen that, in the first instance there is always present the quantity $\log \frac{c - a(n-r)}{c - a(n+r)}$, (101).

and in the second $c - na$. (102).

Inasmuch as the value of t is essentially positive under all real circumstances it follows that the relation determining this condition is that $c - a(n+r)$

shall be positive; that is, $c > a(n+r)$: r is positive; for the first case, and $c > na$ in the second.

In the extreme case when $c = a(n+r)$ or $c = na$, as the situation may be, it will be seen that the body comes to rest at the origin only after an infinite time. When this special realtion is present then our initial equations take on the forms

$$S = a\,e^{-(n+r)t}, \qquad (103).$$

$$S = a\,e^{-nt}. \qquad (104).$$

Further, when $r = n$. $S = \frac{c}{2r}\,e^{-2nt}(2ra - e)$,

$$\text{or} \quad S = e^{-nt}\{a - t(c - an)\}, \qquad (105).$$

it follows that the body will attain the position of equilibrium only when $c = 2na$ and for vâlues of c greater than 2na, it will pass thru zero. Also when $c = 2na + q$, the final position is $S = -q/2n$ and if $c = 2na - q$, the final position is. $S = q/2n$.

As a simple application of an idea or two , and in connection with anequation or two, to be found in the preceeding pages, it will be of interest to consider the effect of a current on the aperiodic motion of a moving magnet.

Let the case be that in which a current of value i acts for a very short time t' on a suspended magnet while at rest in its position of equilibrium. The magnet will receive a positive velocity $c = \frac{uit'}{M}$,

where M is its moment of inertia, u its moment of rotation, that is, the effect of unit current on the magnet. The constants A, B, which appear in the general equation (35) , we find to be $A = -c/2r$, $B = c/2r$.

and the equationn takes the form

$$S = c/2r\left(e^{-(n-r)t} - e^{-(n+r)t}\right) \qquad (106).$$

The magnet reverses its motion at $t_{S=\max} = \frac{1}{2r}\log\frac{n+r}{n-r}$,

and returns asymptotically to the position of rest.

In the case, n = f, the equation becomes simpler. The constants take the values $A = o,\ B = c,$ and the equation appears as $s = c\epsilon^{-nt}t$. (107). The curve of deflections is seen to be concave toward the axis at the origin. its ordinates reach a maximum value of $s_{max} = c/n\epsilon$, at the time $t_{s=max} = 1/n$ and that it has point of inflection at $t_{pi} = 2/n$. The expression for t_{s-max} permits the numerical determination of the current impulse on the magnet to be made.

The curve of velocities begins with an ordinate c at t = o and is seen to be convex toward the time axis. It cuts the same at the time $t_{s=max}$ and it attains at the time t_{pi} a negative maximum and passes thru a point of inflection at $t = 3/n$.

The arithmetical series of times mentioned on page is seen to be present here as well.

Chapter VIII

In the last chapter we were interested with the little problem of the effect of a current impulse on a magnet free to move and at its position of equilibrium. In the present case we shall suppose that the magnet suffers a deflection that becomes constant and that the deflecting current maintains a fixed value.

If the magnet moves under the influence of a constant current of value i. from the zero position to a new position of equilibrium, attained under the united effects of the current and the restoring force, the differential equation in the case becomes

$$\frac{d^2s}{dt^2}+2u\frac{ds}{dt}+f^2s=k, \qquad (107).$$

where k is the new deflecting force divided by the moment of inertia of the system.

The general integral is $s=\epsilon^{-ut}(A\epsilon^{-\lambda t}+B\epsilon^{\lambda t})+k/f^2 \cdot$ (108).

If, when t = 0, s = 0 and s' = 0 then the constants

become $A - K(n-r)/2rf^2$, $B = K(n+r)/2rf^2$.

If we designate by H the horizontal component of the earth's field, and by M the magnetic moment of the magnet and bearing in mind that $f^2 = MH/m$, we find that $k/f^2 = ui/MH$.

Introducing these quantities, our equation is

$$S = \frac{ui}{MH}\left[1 - \frac{e^{-nt}}{2r}\left\{(n+r)e^{rt} - (n-r)e^{-rt}\right\}\right]. \qquad (109).$$

The motion arises, therefore, as was to be expected, from the same laws as in the falling of the magnet, only that in the place of the quantity 'a' we have ui/MH and in accord with the theory, the zero will be attained without oscillations and after an infinite time.

When $n = f$ then our last equation becomes

$$s = ui/MH(1-e^{-nt}(1+nt)). \qquad (110).$$

In addition to the previous combinations of position and velocity, we may suggest the following. Let a current impulse act on the magnet at the instant it starts to mave from the displacement a on its return to zero. Then Eqs. (90) and (90$'$) apply but with c of the positive sign. The magent makes an excursion and asymptotically approaches the zero. If the magnet is already in motion and it receives an impulse at s_1 and t_1, which gives it a velocity of ±c, then there is a discontinuity in its motion. If $n \gtreqless f$ the we have the equations

$$S = \frac{a}{2r}\left\{(n+r)e^{-(n-r)(t_1+t)} - (n-r)e^{-(n+r)(t_1+t)}\right\} \pm c/2r\left\{e^{-(n-r)t} - e^{-(n+r)t}\right\}, \qquad (111),$$

$$S = a\epsilon^{-n(t_1+t)}\left\{1 + n(t_1+t_1)\right\} \pm c\,\epsilon^{-nt}\,t. \qquad (112).$$

Here t is the new time and is counted from the instant of impulse. The right-hand side of Eqs.(111),(112) is the algebraic sum of the right-hand sides of Eqs.(37) and (106) and Eqs. (33$'$) and (107) respectively, only that in the first term $t_1 + t$ stands for t. It is found

just as would be expected, that we have here the superposition of two motions.

If c is negative the zero can be transited and this in case Eq.(111). when $c - \left(ds_1/dt\right) > (n+r)s_1$ and in case Eq.(112) when $c - \left(ds_1/dt\right) > ns_1$. Compare Chapter V.

If the current which has given a constant deflection to a magnet suddenly change from i to i_1, then we have when $n \gtreqless f$, the Eqs.

$$S = \frac{\mu}{MH}\left\{ i_1 + (i_1 - i)\frac{e^{-nt}}{2r}\left\{(n+r)e^{rt} - (n-r)e^{-rt}\right\}\right\}, \quad (108).$$

$$S = \frac{\mu}{MH}\left\{ i_1 + (i - i_1)e^{-nt}(1+nt)\right\}, \quad (109).$$

and the magnet takes up the new position without oscillation.

Chapter X.

In this concluding chapter we shall briefly outline the main features and results of our investigation. In the preceding pages we have been concerned with the study of the damped motion of a suspended body free to execute oscillations under certain conditions and in particular with those cases in which the motion is not oscillatory but aperiodic.

As we have already signified, the treatment applies in all generality to bodies whose return to the initial position of equilibrium is secured by a force that is proportional to the angle of displacement and whose restoration is opposed by a force proportional to the velocity at any instant.

Bodies having a single torsion suspension satisfy the first requirement and for all values of the angular deflection. Suspended bodies having

torsionless supports must clearly secure their restoring force from agencies outside the moving system itself; as in the magnet suspended by a long, thin, silk fiber, the earth's magnetic field being the agency. In this case the force in mind is strictly proportional to the sine of the displacement and hence it can only be considered as meeting the conditions assumed when the deflection is so small thatthe sine is sensibly equal to its angular magnitude.

From the above we see that the subject in hand applies to the moving system of a d'Arsonval galvanometer, with torsion suspension, and to the Thomson galvanometer, with moving magnet system, and torsionless suspension, provided the angular deflection is small.

The energy of the system obviously remains constant if there be no damping forces, as the oscillations continue without end and without diminution.

Damping agencies are energy-consuming and it is thru their presence that the energy of the system is transformed from a motional to a non-motional character or transferred to other bodies, or both.

In the actual, material conditions of the above types that may arise, the damping owes its existence to eddy-current reaction or air-friction. The former is rigidly proportional to the first power of the velocity and the second is quite approximately so for low velocities. The actual law of the relationship in the second case is very irregular, varying with the angular velocity and with the surface exposed to air contact, if the speed is kept constant.

Accordingly it is seen that the equations herein deduced may be taken as applying directly to galvanometers of the above-mentioned types, for in the one case we generally have a torsion suspension and

eddy-current damping and in the other, earth's field control and air damping.

Passing now to the curves in the text, we easily recognize that on Sheet I as periodic motion heavily damped—as that of a pendulum swinging in a tank of water. The coefficient of damping is $2n = .48$, $n = .24$, the coefficient of restitution is $f^2 = 2.41$, $f = 1.55$ and $f > n$ obviously.

Aperiodic motion, defined by the condition that $n > f$, is shown on Sheet II. The maximum initial displacement, a, was taken as 100 in every case and in general the other constants were so chosen that in 20 units of time, the angular position had decreased to about 5. Curves Nos. 1, 2, ($n = .237$, $f = .232$, $r = .05$) illustrate the manner in which the displacement changes when the restoring forces are in the ratio of .014 to .054, or 1 to 4 approximately. Curves Nos. 3, 4. corresponding to the same condition as that

represented by No.2, give the velocity and the acceleration; the former starting with zero value, attains a negative maximum at $t_{max} = 1/2r \log(n+r / n-r) = 4.187$ and then falls off to zero at infinite time. The acceleration starts in with a positive value of 5.38, becomes zero at $t = 4.187$ and then remaining negative (shown above the axis for convenience) to the end of time but attaining a maximum in its slowly diminishing career.

Sheet IV. Curves Nos. 1,2 represent the deflections-time relationship for aperiodic motion with initial velocity. The specifications are $n = r = .24$, initial velocity $+c, -c = \pm 47.4$ Curves 1,3,6 form one group and 2,4,5 the other. All motion continues until $t = \infty$, tho it appears finished at $t = 10$.

Sheet III. Aperiodic motion with critical damping. $n = f = .237$ In this case we have the border-

line of oscillatory and aperiodic motions. If $n > f$ there is no oscillation and if $n < f$ oscillation takes place. Critical damping insures the speediest return to zero. Theoretically, all aperiodic motion is not completed until after an infinite time but in the practical case, critically damping may secure a return ot zero that is for all purposes complete in a short time. In this particular instance, the velocity is negative, attains a maximum at $1/n = 4.22$ and then decreases to zero at infinite time. The acceleration is positive at the start, changes sign at $1/n$ and then remains negative for the rest of time tho rising to a maximum at $2/n = 8.44$.

These three curves are closely similiar to those numbered 2,3,4. on Sheet II. This arises from the fact f in these last curves is so nearly equal to n. the condition being very nearly that of critical damping,

and inadvertantly used to secure a curve of deflections of decided character. If however curve No.1 of SheetII be compared with No.1 of Sheet III, the effect of critical damping is very clearly seen.

Sheet V. This curve in two parts gives the true damping factor for galvanometer or other motions coforming to the assumptions herein laid down. For undamped motion, k, the ratio of two consecutive swings, is unity and increases to the value of 2.718 when oscillatory motion is changed to aperiodic. and for the case of critical damping. This curve is available for practical purposes and gives the correct values of the undamped swings, a character not possessed by any other formula than the one plotted here, namely.

$S_0 = S_1 k^{\frac{1}{\pi}\tan^{-1}\frac{\pi}{\lambda}}$.

Sheet VI. We have here the constantly positive,per cent error curve of the damping factor given in the approx-

imate form $(s_1 / s_3)^{1/4}$.

Sheet VII. A curve exhibiting the per cent error introduced thru the use of the ordinary, approximate damping factor $(1 + \lambda/2)$. The error is positive when $k < 2.37$ and negative $k > 2.37$. There is a maximum positive error at $k = 2$ approximately and zero error at $k = 2.37$. It is to be noted that the error is for all values of k that would be encountered in common practice, very small and therefore negligible.

Sheet VIII. A curve showing the per cent error present in the use of the damping factor in the form $(s_1 - s_3)^{1/4}$ Even if the value of the constant k be small the error is considerable and quickly rises to values that render its use out of the question.

Sheet IX. Displacement and velocity curves for aperiodic motion with initial velocity = .5

Note.

If the expression with which we have been concerned in the preceeding paper are desired to apply specifically to a material body of moment of inertia K and possessing a directive or restoring force D, then our familiar constant f^2 is equal to D/K. If the factor with which the instantaneous angular velocity must be multiplied to give the moment of the restoring force be p, then $2n = p/K$.

The general equation when referring to this particular application is usually given in the form

$$\frac{d^2s}{dt^2} + \frac{h}{K}\frac{ds}{dt} + \frac{D}{K}\,s = 0,$$

When the motion of a suspended magnet, of magnetic moment M and moment of inertia and in a magnetic field of strength H is under consideration, then $f^2 = MH/K$, and $2n = p/K$ as before. Accordingly, we have

$$\frac{d^2s}{dt^2} + \frac{k}{K}\frac{ds}{dt} + \frac{MHs}{K} = 0.$$

Or if it is an electric circuit phenomenon that is to be the subject for investigation, the resistance, capacity and inductance being designated by R, C, and L, respectively, then our potent equation takes the form

$$\frac{d^2s}{dt^2} + \frac{R}{L}\frac{ds}{dt} + \frac{1}{LC}s = 0, \text{ or}$$

since the variable is the current, it may be more correctly written as

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0.$$

We therefore have $2n = R/L \qquad f^2 = 1/LC$

$\qquad\qquad\qquad\qquad = p/K \qquad\quad = D/K$

and it is seen that $p \equiv R$, $K \equiv L$, $D \equiv 1/C$.

In words, the restoring force D, the elasticity of the system, finds its duality in the reciprocal of the capacity. Conversely, it is seen that capacity corresponds to the compliancy in mechanical notions. The moment of inertia K finds its existence in the electric circuit in the inductance thereof. And lastly, the familiar coefficient of damping, 2n, occurs in another dress in the form R/L.